International Journal of Scientific & Engineering Research, Volume11, Issue3, March-2020
ISSN 2229-5518

Simple and Better Approach to Analysis of Plates with Large Deflection

[1]O. M. Ibearugbulem, [2]J. C. Ezeh, [3]U. C. Anya, [4]C. P. Anyadiegwu
[1,2,3] Department of Civil Engineering, Federal University of Technology Owerri, Nigeria
[4] Department of Civil Engineering, Federal Polytechnic Nekede, Nigeria. Chibuezepius52@gmail.com

Abstract— this paper presents a new, simple and better approach to analyze thin rectangular plates with large deflection. This fit is achieved by circumventing the use of Airy's stress function. It was possible to circumvent the Airy's stress function solving the Von-Karman strain-displacement equation in closed form. The resulting non-linear strain-deflection equations were substituted into the constative equations to obtain the non-linear stress-deflection equations. These stresses and strains were used to obtain the total potential energy functional of a rectangular plate with large deflection. This functional was minimized with respect to displacement to obtain one governing equation and two compatibility equations, which were solved to obtain the in-plane displacements in terms of deflection. The energy functional was again minimized with respect to the coefficient of deflection to obtain the various formula for analyzing the plate with large deflection. The values of load parameters calculated were compared with those obtained by Levy and the recorded maximum percentage difference is 2.39%. This validates the present method. From the results, it is concluded that for ratios of deflection-to-thickness (w/t) of the plate less than 0.35 small deflection theory can be used with up to 90% accuracy level. When w/t is up to or more than 0.35 large deflection theory is recommended for accuracy.

Index Terms— Airy's stress functions, Von-Karman, strain-displacement, stress-displacement, total potential energy functional, minimize, load parameter.

NOTATIONS

A	Area of thin plate
a	Length of the primary dimension of the plate
b	Length of the secondary dimension of the plate
h	Shape (profile) function
u	Displacement of plate in x -direction
v	Displacement of plate in y –direction
V_{NX}	External axial work
ε_{xxm}	Middle surface strain component in x-direction
ε_{yym}	Middle surface strain component in y-direction
Δ	Deflection coefficient
γ	Shear strain of the plate
γ_{xy}	Shear strain on x-y plane
D	Flexural rigidity of the plate
E	Young modulus of elasticity of isotropic plate
S	Non dimension axis (quantity) parallel to z axis
Z	Cross section modulus
ε_{xx}	Strain component in x-direction
ε_{yy}	Strain component in y-direction
M	Moment
R	Non dimension axis (quantity) parallel to x axis
Q	Non dimension axis (quantity) parallel to y axis
σ_b	Bending stress
t	Thickness of the plate
c_1	Unknown constant
N_x	Inplane force of a plate perpendicular to x-direction
U_m	Membrane strain energy
v_o	Middle surface displacement of plate in y-direction
q	Lateral load on plate
σ_m	Membrane stress
σ_x	Normal stresses acting in x directions
w	Displacement of plate in z-direction
u_o	Middle surface displacement of plate in x-direction
σ_y	Normal stresses acting in y directions
τ	Shear stress of the plate
τ_{xy}	Shear stress on x-y plane
Π	Total potential energy of the plate
β	Aspect ratio that is, $\beta = b/a$
R	$= x/a$
Q	$= y/b$
S	$= z/t$

ISSN 2229-5518

1 Introduction

Razdolsky[1] stated it clearly that analytical solution of von Karman nonlinear equations is unattainable. This stand is corroborated by Zhang et al. [7] and Rezaiee-Pajand and Estiri [10]. Von-Karman type of nonlinear strain-displacement relation "[9]" governs most works on analyses of rectangular plate with large deflection. This Von-Karman type of strain-displacement relation is made of two parts: Linear part (Kirchhoff's strain-displacement, taken herein as bending strain-displacement) and the non-linear part (membrane strain-displacement). The plane Von-Karman strain-displacements relations "[9], [12]" are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u_0}{\partial x}\right] \quad (1)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + \left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial v_0}{\partial y}\right] \quad (2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x\,\partial y} + \left[\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right] \quad (3)$$

Difficulty encountered in handling the membrane part of the Von-Karman strain-displacement constitutes the problem of analysis of plates with large deflection. This problem is mainly resting on obtaining mathematical expressions for the membrane in-plane displacements u_0 and v_0.

Earlier works on analyses of rectangular plate with large deflection assume expressions for u_0 and v_0. This makes their final results not to be taken as exact results [14], [8], [3], [2], [4], [16], and [5]. In the process of solving the Von-Karman equations, stress function (usually reffered to as Airy's stress function) is used. As in the case of in-plane membrane displacement, earlier scholars usually assume expression for the Airy's stress function, which renders the results from their approximate [8], [4], and [5]. Determination of exact expression for the Airy's stress function seems intractable. However, Oguaghamba [11], Onodagu [13] and Enem [6] in the PhD works determined polynomial expressions for Airy's stress function. They accomplished this fit by integrating the governing equation and compatibility equation of the plate. However, the expressions for Airy's stress functions determined by these scholars are quite lengthy and encompassing. Mere looking at these expressions can discourage an analyst.

These problems adduced herein is the propelling factor for the present research. The tacit for this research is "can Von-Karman strain – displacement relations be used in the analysis rectangular plates with large deflection without using or introducing Airy's stress function?" This means circumvention of Airy's stress function in the analysis of rectangular plates with large deflection.

- *Anyadiegwu P.C is currently pursuing PhD degree program in structural engineering in Federal University Technology, Owerri, and FUTO. E-mail: chibuezepius52@gmail.com*
- *Engr. Prof. J.C. Ezeh, professor of Structural engineering in Federal University Technology, Owerri. E-mail:*

2.1 Membrane in-plane displacements

The membrane strain –displacement of rectangular plate with large deflection is obtained from "(1)," "(2)," and "(3),". They are the terms in the square brackets: numbe:

$$\varepsilon_{xxm} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u_0}{\partial x} \quad (4)$$

$$\varepsilon_{yym} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial v_0}{\partial y} \quad (5)$$

Minimizing "(4)," and "(5)," with respect to $\left(\frac{\partial}{\partial x}\right)$ and $\left(\frac{\partial}{\partial y}\right)$ respectively gives:

$$\frac{\partial(\varepsilon_{xxm})}{\partial\left(\frac{\partial}{\partial x}\right)} = \frac{1}{2} * \frac{\partial}{\partial\left(\frac{\partial}{\partial x}\right)}\left[\left(\frac{\partial}{\partial x}\right)^2 * w^2\right] + \frac{\partial}{\partial\left(\frac{\partial}{\partial x}\right)}\left[\frac{\partial}{\partial x} * u_0\right] = \frac{1}{2}\frac{\partial w^2}{\partial x} + u_0 = 0 \quad (6)$$

$$\frac{\partial(\varepsilon_{yym})}{\partial\left(\frac{\partial}{\partial y}\right)} = \frac{1}{2} * \frac{\partial}{\partial\left(\frac{\partial}{\partial y}\right)}\left[\left(\frac{\partial}{\partial y}\right)^2 * w^2\right] + \frac{\partial}{\partial\left(\frac{\partial}{\partial y}\right)}\left[\frac{\partial}{\partial y} * v_0\right] = \frac{1}{2}\frac{\partial w^2}{\partial y} + v_0 = 0 \quad (7)$$

Rearranging "(6),"and "(7)," gives:

$$u_0 = -\frac{1}{2}\frac{\partial w^2}{\partial x} \quad (8)$$

$$v_0 = -\frac{1}{2}\frac{\partial w^2}{\partial y} \quad (9)$$

The coefficients of u_0 and v_0 (in equations 8 and 9) that makes the membrane strains (equations 4 and 5) zero is minus half. Hence, to avoid zero membrane strains (which violets the basic assumption of large deflection theory) other constants that are not minus half is used. However, it shall be pertinent to optimize the constant. This optimum value of the constant will enable the plate carry loads with its membrane strength when it looses its bending strength. This optimum value of the constant was obtained by Ibearugbulem [12] in his unpublished class note. Following his work, the minus half in "(8)," and "(9)," are replaced with unknown constant c_1 as:

$$u_0 = c_1\frac{\partial w^2}{\partial x} \quad (10)$$

$$v_0 = c_1\frac{\partial w^2}{\partial y} \quad (11)$$

Substituting "(7)," and "(8)," into "(4)," and "(5)," gives:

$$\varepsilon_{xxm} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + c_1\left(\frac{\partial w}{\partial x}\right)^2 = c_2\left(\frac{\partial w}{\partial x}\right)^2 \quad (12)$$

$$\varepsilon_{yym} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + c_1\left(\frac{\partial w}{\partial y}\right)^2 = c_2\left(\frac{\partial w}{\partial y}\right)^2 \quad (13)$$

International Journal of Scientific & Engineering Research, Volume 11, Issue 3, March-2020
ISSN

$$Where: \quad c_2 = c_1 + \frac{1}{2} \tag{14}$$

The common definition of bending stress is:

$$\sigma_b = \frac{M}{Z} \tag{15}$$

M and Z are moment and cross section modulus (first moment of area) respectively. Cross section modulus of a rectangular cross section is $bt^2/6$. Where b and t are the width and thickness of the section respectively. Substituting this into equation 15 gives:

$$\sigma_b = \frac{6M}{bt^2} \tag{16}$$

Consider the moment of a continuum by in-plane load and deflection shown on Figure 1.

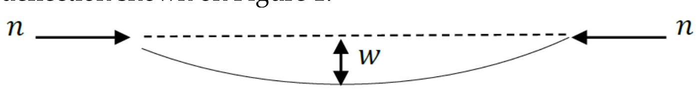

Figure 1:A deflected continuum from inplane force

From Figure 1, moment is given as:

$$M = n\,.t = (N_x bt)t = N_x bt^2 \tag{17}$$

Substituting equation 17 into equation 16 gives:

$$\sigma_b = \frac{6N_x bt^2}{bt^2} = 6N_x \tag{18}$$

During large deflection, the plate can lose its bending strength and rely only on membrane strength. At this point, the entire bending stress, σ_b transforms to membrane stress, σ_m. that is equation 18 becomes the membrane stress equation:

$$\sigma_{mx} = 6N_x \tag{19}$$

Membrane strain energy is obtained using the membrane strain and membrane stress, "(12)," and "(19)," gives:

$$U_m = \frac{1}{2}\int\int\int \varepsilon_{xxm}\,\sigma_{mx}\,dxdydz$$

$$U_m = \frac{1}{2}\int\int\int c_2\left(\frac{\partial w}{\partial x}\right)^2 \times 6N_x\ dxdydz.$$

That is:

$$U_m = \frac{6c_2N_x}{2}\int\int\int\left(\frac{\partial w}{\partial x}\right)^2 dxdydz \tag{20}$$

At any arbitrary point on the continuum, the external axial work is given as:

$$V_{N_x} = \frac{N_x}{2}\int\int\int\left(\frac{\partial w}{\partial x}\right)^2 dxdydz \tag{21}$$

Subtracting "(21)," from "(20)," and minimizing the resulting functional with respect to deflection (w) gives:

$$c_2 = \frac{1}{6} \tag{22}$$

Substituting "(22)," into "(14)," and rearranging gives:

$$c_1 = -\frac{1}{3} \tag{23}$$

Substituting "(23)," into "(10)," and "(11)," gives:

$$u_0 = -\frac{1}{3}\frac{\partial w^2}{\partial x} \tag{24}$$

$$v_0 = -\frac{1}{3}\frac{\partial w^2}{\partial y} \tag{25}$$

2.2 Displacements and nonlinear kinematics of rectangular plates with large deflection

Substituting "(24)" and "(25)" into "(1)" and"(2)" gives:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \frac{1}{6}\left(\frac{\partial w}{\partial x}\right)^2 \tag{26}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + \frac{1}{6}\left(\frac{\partial w}{\partial y}\right)^2 \tag{27}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x\,\partial y} + \frac{1}{3}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$$

$$= 2\left[-z\frac{\partial^2 w}{\partial x\partial y} + \frac{1}{6}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \tag{28}$$

Integrating "(26)," and "(27)," with respect to x and y respectively gives:

$$u = -z\frac{\partial w}{\partial x} + \frac{1}{6}\frac{\partial w^2}{\partial x} \tag{29}$$

$$v = -z\frac{\partial w}{\partial y} + \frac{1}{6}\frac{\partial w^2}{\partial y} \tag{30}$$

2.3 Constitutive relations of rectangular plates with large deflection

The constitutive relations of a rectangular plate with plane stress assumption are:

$$\sigma_{xx} = \frac{E}{(1-\mu^2)}\left(\varepsilon_{xx} + \mu\varepsilon_{yy}\right) \tag{31}$$

$$\sigma_{yy} = \frac{E}{(1-\mu^2)}\left(\mu\varepsilon_{xx} + \varepsilon_{yy}\right) \tag{32}$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy} = \frac{E(1-\mu)}{2(1-\mu^2)}\gamma_{xy} \tag{33}$$

Substituting "(26)," "(27)," and "(28)," into "(31)," "(32)," and "(33)," gives the stress-deflection relations:

$$\sigma_{xx} = \frac{E}{(1-\mu^2)}\left(-z\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] + \frac{1}{6}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \mu\left(\frac{\partial w}{\partial y}\right)^2\right]\right) \tag{34}$$

$$\sigma_{yy} = \frac{E}{(1-\mu^2)}\left(-z\left[\mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] + \frac{1}{6}\left[\mu\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\right) \tag{35}$$

$$\tau_{xy} = \frac{E(1-\mu)}{(1-\mu^2)} * \left[-z\frac{\partial^2 w}{\partial x\partial y} + \frac{1}{6}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \tag{36}$$

International Journal of Scientific & Engineering Research, Volume 11, Issue 3, March-2020
ISSN

2.4 Total potential energy functional of rectangular plate with large deflection

The total potential energy functional of a classical rectangular plate in pure bending is given as:

$$\Pi = \frac{1}{2}\iiint (\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \tau_{xy}.\gamma_{xy})\ dx\,.dy\,.dz - -q\iint w\,dx\,dy \qquad (37)$$

Substituting the stress-deflection relations "(34)," "(35)," and "(36)," and strain-deflection relations "(26)," "(27)," and "(28)," into "(37)," and carrying out the closed domain integration with respect to z coordinate gives:

$$\Pi = \frac{D}{2}\iint\left\{\left[\left(\frac{d^2w}{dx^2}\right)^2 + 2\left(\frac{d^2w}{dxdy}\right)^2 + \left(\frac{d^2w}{dy^2}\right)^2\right] + \frac{g}{36}\left[\left(\frac{\partial w}{\partial x}\right)^4 + 2\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^4\right] - \frac{2}{D}qw\right\}dx\,.dy \qquad (38)$$

$$Where{:}\ D = \frac{\mathrm{Et}^3}{12(1-\mu^2)};\ g = \frac{12}{\mathrm{t}^2} \qquad (39)$$

Writing equation 38 in terms of the non-dimensional coordinates (R = x/a and Q = y/b, S = z/t, where: a and b are plate lengths along x and y-axes) in closed domain gives:

$$\Pi = \frac{bD}{2a^3}\int_0^1\int_0^1\left\{\left[\left(\frac{d^2w}{dR^2}\right)^2 + \frac{2}{\beta^2}\left(\frac{d^2w}{dRdQ}\right)^2 + \frac{1}{\beta^4}\left(\frac{d^2w}{dQ^2}\right)^2\right] + \frac{g}{36}\left[\left(\frac{\partial w}{\partial R}\right)^4 + \frac{2}{\beta^2}\left(\frac{\partial w}{\partial R}\right)^2\left(\frac{\partial w}{\partial Q}\right)^2 + \frac{1}{\beta^4}\left(\frac{\partial w}{\partial Q}\right)^4\right] - 2\frac{qa^4}{D}w\right\}dR\,dQ \qquad (40a)$$

Without loss of generality,"(40a)" is rewritten as:

$$\Pi = \frac{bD}{2a^3}\int_0^1\int_0^1\left\{\left[\left[\frac{d^3}{dR^3}\frac{dw^2}{dR} + \frac{1}{\beta^2}\frac{d^3}{dRdQ^2}\frac{dw^2}{dR} + \frac{1}{\beta^2}\frac{d^3}{dR^2dQ}\frac{dw^2}{dQ} + \frac{1}{\beta^4}\frac{d^3}{dQ^3}\frac{dw^2}{dQ}\right]\right] + \frac{g}{36}\left[\frac{\partial^2}{\partial R^2}\left(\frac{\partial w^2}{\partial R}\right)^2 + \frac{1}{\beta^2}\frac{\partial}{\partial R}.\left(\frac{\partial w^2}{\partial R}\right).\frac{\partial}{\partial Q}.\left(\frac{\partial w^2}{\partial Q}\right) + \frac{1}{\beta^4}\frac{\partial^2}{\partial Q^2}\left(\frac{\partial w^2}{\partial Q}\right)^2\right] - 2\frac{qa^4}{D}w\right\}dR\,dQ \qquad (40b)$$

$$\Pi = \frac{bD}{2a^3}\int_0^1\int_0^1\left\{\left[-3\left[\frac{d^3}{dR^3}u_0 + \frac{1}{\beta^2}\frac{d^3}{dRdQ^2}u_0 + \frac{1}{\beta}\frac{d^3}{dR^2dQ}v_0 + \frac{1}{\beta^3}\frac{d^3}{dQ^3}v_0\right]\right] + \frac{9g}{36}\left[\frac{\partial^2}{\partial R^2}{u_0}^2 + \frac{1}{\beta}\frac{\partial}{\partial R}.u_0.\frac{\partial}{\partial Q}.v_0 + \frac{1}{\beta^2}\frac{\partial^2}{\partial Q^2}{v_0}^2\right] - 2\frac{qa^4}{D}w\right\}dR\,dQ \qquad (40c)$$

Substituting "(24)," and "(25)," into "(40b),"gives:

2.5 Governing and compatibility equations of rectangular plate with large deflection

Minimization of the total energy functional with respect to deflection, w gives the governing equation. Minimizing the total energy functional with respect to in-plane displacements u_0 and v_0 gives the compatibility equations along x and y axes respectively. Minimizing equation 40a with respect w gives:
Rearranging "(41a)," gives the governing equation:

$$\frac{\partial \Pi}{\partial w} = \frac{bD}{2a^3}\int_0^1\int_0^1\left\{\left[2\frac{d^4w}{dR^2} + \frac{4}{\beta^2}\frac{d^4w}{dR^2dQ^2} + \frac{2}{\beta^4}\frac{d^4w}{dQ^4}\right] + \frac{g}{36}\left[4\frac{\partial^4w^3}{\partial R^4} + \frac{8}{\beta^2}\frac{\partial^4w^3}{dR^2dQ^2} + \frac{4}{\beta^4}\frac{\partial^4w^3}{\partial Q^4}\right] - 2\frac{qa^4}{D}w\right\}dR\,dQ = 0 \qquad (41a)$$

$$\frac{\partial \Pi}{\partial w} = \frac{bD}{2a^3}\int_0^1\int_0^1\left\{\left[2\frac{\partial^4w}{\partial R^2} + \frac{4}{\beta^2}\frac{\partial^4w}{\partial R^2dQ^2} + \frac{1}{\beta^4}\frac{\partial^4w}{\partial Q^4}\right] + \frac{g}{36}\left[4\frac{\partial^3w}{\partial R^3}\left(\frac{\partial w^2}{\partial R}\right) + \frac{4}{\beta^2}\frac{\partial^3w}{\partial R\partial Q^2}\left(\frac{\partial w^2}{\partial R}\right) + \frac{4}{\beta^2}\frac{\partial^3w}{dR^2\partial Q}\left(\frac{\partial w^2}{\partial Q}\right) + \frac{4}{\beta^4}\frac{\partial^3w}{\partial Q^3}\left(\frac{\partial w^2}{\partial Q}\right)\right] - 2\frac{qa^4}{D}w\right\}dR\,dQ = 0 \qquad (41b)$$

Minimizing "(40c)," with respect to u_0 gives:

$$\frac{\partial \Pi}{\partial u_0} = \frac{bD}{2a^3}\int_0^1\int_0^1\left\{-3\left[\frac{d^3}{dR^3} + \frac{1}{\beta^2}\frac{d^3}{dRdQ^2}\right] + \frac{9g}{36}\left[2\frac{\partial^2}{\partial R^2}u_0 + \frac{1}{\beta}\frac{\partial}{\partial R}.\frac{\partial}{\partial Q}.v_0\right]\right\}dR\,dQ = 0 \qquad (42)$$

Minimizing "(40c)," with respect to v_0 gives:

$$\frac{\partial \Pi}{\partial v_0} = \frac{bD}{2a^3}\int_0^1\int_0^1\left\{-3\left[\frac{1}{\beta}\frac{d^3}{dR^2dQ} + \frac{1}{\beta^3}\frac{d^3}{dQ^3}\right] + \frac{9g}{36}\left[\frac{1}{\beta}\frac{\partial}{\partial R}.u_0.\frac{\partial}{\partial Q} + \frac{2}{\beta^2}\frac{\partial^2}{\partial Q^2}v_0\right]\right\}dR\,dQ = 0 \qquad (43)$$

Simplifying "(42),"and "(43),"respectively gives the two compatibility equations:

$$-\left[\frac{\partial^2}{\partial R^2} + \frac{1}{\beta^2}\frac{\partial^2}{\partial Q^2}\right] + \left[\frac{g}{6}\frac{\partial}{\partial R}u_0 + \frac{g}{12\beta}\frac{\partial}{\partial Q}.v_0\right] = 0 \qquad (44)$$

$$-\left[\frac{\partial^2}{\partial R^2} + \frac{1}{\beta^2}\frac{\partial^2}{\partial Q^2}\right] + \left[\frac{g}{12}\frac{\partial}{\partial R}.u_0 + \frac{g}{6\beta}\frac{\partial}{\partial Q}v_0\right] = 0 \qquad (45)$$

Solving "(44)," and "(45)," simultaneously gives:

$$u_0 = \frac{8}{g}\frac{\partial}{\partial R} \qquad (46)$$

$$v_0 = \frac{8}{g\beta}\frac{\partial}{\partial Q} \qquad (47)$$

International Journal of Scientific & Engineering Research, Volume 11, Issue 3, March-2020
ISSN

Substituting "(24)," and "(25)," into "(46)," and "(47)," respectively rearranging gives:

$$\frac{\partial w^2}{\partial R} = -\frac{24}{g}\frac{\partial}{\partial R} \tag{48}$$

$$\frac{\partial w^2}{\partial Q} = -\frac{24}{g}\frac{\partial}{\partial Q} \tag{49}$$

Substituting "(48),"and "(59)," into "(41b)," and simplifying gives:

$$\frac{\partial^4 w}{\partial R^4} + \frac{2}{\beta^2}\frac{\partial^4 w}{\partial R^2 \partial Q^2} + \frac{1}{\beta^4}\frac{\partial^4 w}{\partial Q^4} + \frac{3qa^4}{D}w = 0 \tag{50}$$

The ready solution to "(50),"in trigonometric form is: In trigonometric form, solution to "(40),"is:

$$w = [a_0 \quad a_1 \quad a_2 \quad a_3]\begin{bmatrix} 1 \\ R \\ \cos kR \\ \sin kR \end{bmatrix} \times [b_0 \quad b_1 \quad b_2 \quad b_3]\begin{bmatrix} 1 \\ Q \\ \cos gQ \\ \sin gQ \end{bmatrix} \tag{51}$$

In a denotational form "(51)" becomes:

$$w = a_i h_x \times b_i h_y = Ah \tag{52}$$

Formulas form analyzing rectangular plate with large deflection

Substituting "(52)," into "(32)," gives:

$$\Pi = \frac{A^2 bD}{2a^3}\int_0^1\int_0^1 \left[\left(\frac{d^2h}{dR^2}\right)^2 + \frac{2}{\beta^2}\left(\frac{d^2h}{dRdQ}\right)^2 + \frac{1}{\beta^4}\left(\frac{d^2h}{dQ^2}\right)^2\right] dR\, dQ + \frac{A^4 \mathrm{bgD}}{36 \times 2a^3}\int_0^1\int_0^1 \left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{\beta^2}\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 + \frac{1}{\beta^4}\left(\frac{\partial h}{\partial Q}\right)^4\right] dR\, dQ - Aabq \iint w\, dR\, dQ \tag{53a}$$

In a denotational form, "(53a),"becomes:

$$\Pi = \frac{A^2 bD}{2a^3}\left[k_{bRR} + \frac{2k_{bRQ}}{\beta^2} + \frac{k_{bQQ}}{\beta^4}\right] + \frac{A^4 \mathrm{bgD}}{72a^3}\left[k_{mRR} + \frac{2k_{mRQ}}{\beta^2} + \frac{k_{mQQ}}{\beta^4}\right] - Aabq\, k_q \tag{53b}$$

Where:

$$k_{bRR} = \int_0^1\int_0^1 \left(\frac{d^2h}{dR^2}\right)^2 dR\, dQ;\; k_{bRQ} = \int_0^1\int_0^1 \left(\frac{d^2h}{dR\, dQ}\right)^2 dR\, dQ;$$

$$k_{bQQ} = \int_0^1\int_0^1 \left(\frac{d^2h}{dQ^2}\right)^2 dR\, dQ;\; k_{mRR} = \int_0^1\int_0^1 \left(\frac{\partial h}{\partial R}\right)^4 dR\, dQ;$$

$$k_{mRQ} = \int_0^1\int_0^1 \left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 dR\, dQ;\; k_{mQQ} = \int_0^1\int_0^1 \left(\frac{\partial h}{\partial Q}\right)^4 dR\, dQ$$

Minimizing "(53b)," with respect to coefficient of deflection, A gives:

$$\frac{\partial \Pi}{\partial \mathrm{A}} = \frac{AbD}{a^3}\left[k_{bRR} + \frac{2k_{bRQ}}{\beta^2} + \frac{k_{bQQ}}{\beta^4}\right] + \frac{A^3 \mathrm{bgD}}{18a^3}\left[k_{mRR} + \frac{2k_{mRQ}}{\beta^2} + \frac{k_{mQQ}}{\beta^4}\right] - abq\, k_q = 0 \tag{54}$$

Simplifying and writing "(54)," in denotational form gives:

$$A\, k_{bT} + \frac{A^3 \mathrm{g}}{18}(k_{mT}) - \frac{qa^4}{D}k_q = 0 \tag{55}$$

$$Where: \quad k_{bT} = k_{bRR} + \frac{2}{\beta^2}k_{bRQ} + \frac{1}{\beta^4}k_{bQQ};$$

$$k_{mT} = k_{mRR} + \frac{2}{\beta^2}k_{mRQ} + \frac{1}{\beta^4}k_{mQQ} \tag{56}$$

Substituting equations for g and D from "(39)" into "(55)" and dividing through by t gives:

$$\left(\frac{A}{t}\right) k_{bT} + \frac{2}{3}\left(\frac{A}{t}\right)^3 k_{mT} - 12(1-\mu^2)\frac{q}{E}\left(\frac{a}{t}\right)^4 k_q = 0 \tag{57}$$

Rearranging "(57)" gives:

$$\left(\frac{A}{t}\right)^3 + \frac{3}{2}\left(\frac{A}{t}\right)\frac{k_{bT}}{k_{mT}} - 18(1-\mu^2)\frac{q}{E}\left(\frac{a}{t}\right)^4 \frac{k_q}{k_{mT}} = 0 \tag{58}$$

Simplifying "(58)" gives:

$$\Delta^3 + \theta\Delta + c = 0 \tag{59}$$

$$Where: \Delta = \left(\frac{A}{t}\right);\; \theta = \frac{3}{2}\frac{k_{bT}}{k_{mT}};\, c = -18(1-\mu^2)\frac{q}{E}\left(\frac{a}{t}\right)^4 \frac{k_q}{k_{mT}} \tag{60}$$

Solving "(59)," gives the real root as:

$$\Delta = -\frac{\theta \times (2/3)^{1/3}}{[-9c + 3^{1/2} \times [4\theta^3 + 27c^2]^{1/2}]^{1/3}} + \frac{[-9c + 3^{1/2} \times [4\theta^3 + 27c^2]^{1/2}]^{1/3}}{2^{1/3} \times 3^{2/3}} \tag{61a}$$

$$\Delta = -\frac{\Delta_{11}}{\Delta_{12}} + \frac{\Delta_{12}}{\Delta_{22}} \tag{61b}$$

$$Where: \Delta_{11} = \theta \times (2/3)^{1/3};\; \Delta_{12} = [-9c + 3^{1/2} \times [4\theta^3 + 27c^2]^{1/2}]^{1/3};\; \Delta_{22} = 2^{1/3} \times 3^{2/3}$$

Thus, the coefficient of deflection of plate with large deflection is:

$$\mathrm{A} = \Delta \mathrm{t} \tag{62}$$

Substituting "(62)," into "(52)," gives:

$$\mathrm{w} = \Delta \mathrm{th} \tag{63}$$

Substituting "(63)," into (29), (30), (34), (35) and (36) and writing them in terms non-dimensional coordinates gives:

$$u = \frac{\Delta t^2}{a}\left[-S\frac{\partial h}{\partial R} + \frac{\Delta}{6}\frac{\partial h^2}{\partial R}\right] \tag{64}$$

International Journal of Scientific & Engineering Research, Volume 11, Issue 3, March-2020
ISSN

$$v = \frac{\Delta t^2}{\alpha\beta}\left[-S\frac{\partial h}{\partial Q} + \frac{\Delta}{6}\frac{\partial h^2}{\partial Q}\right] \qquad (65)$$

$$\sigma_{xx} = \frac{E\Delta t^2}{(1-\mu^2)a^2}\left(-S\left[\frac{\partial^2 h}{\partial R^2} + \frac{\mu}{\beta^2}\frac{\partial^2 h}{\partial Q^2}\right] + \frac{\Delta}{6}\left[\left(\frac{\partial h}{\partial R}\right)^2 + \frac{\mu}{\beta^2}\left(\frac{\partial h}{\partial Q}\right)^2\right]\right) \qquad (66)$$

$$\sigma_{yy} = \frac{E\Delta t^2}{(1-\mu^2)a^2}\left(-S\left[\mu\frac{\partial^2 h}{\partial R^2} + \frac{1}{\beta^2}\frac{\partial^2 h}{\partial Q^2}\right] + \frac{\Delta}{6}\left[\mu\left(\frac{\partial h}{\partial R}\right)^2 + \frac{1}{\beta^2}\left(\frac{\partial h}{\partial Q}\right)^2\right]\right) \qquad (67)$$

$$\tau_{xy} = \frac{E(1-\mu)\Delta t^2}{(1-\mu^2)a^2} * \frac{1}{\beta}\left[-S\frac{\partial^2 h}{\partial R \partial Q} + \frac{\Delta}{6}\left(\frac{\partial h}{\partial R}\right)\left(\frac{\partial h}{\partial Q}\right)\right] \qquad (68)$$

The load parameter is obtained from "(57),"by making it the subject (load parameter):

$$\frac{q}{E}\left(\frac{a}{t}\right)^4 = \frac{(A/t)}{12(1-\mu^2)}\frac{k_{bT}}{k_q} + \frac{(A/t)^3}{18(1-\mu^2)}\frac{k_{mT}}{k_q} \qquad (69)$$

For small deflection theorem, the equation of deflection coefficient parameter, A/t is obtained from "(57)," by considering the membrane part to be zero. That is:

$$\left(\frac{A}{t}\right) k_{bT} - 12(1-\mu^2)\frac{q}{E}\left(\frac{a}{t}\right)^4 k_q = 0 \qquad (70)$$

Making the deflection coefficient parameter, A/t the subject gives:

$$\left(\frac{A}{t}\right) = 12(1-\mu^2)\frac{q}{E}\left(\frac{a}{t}\right)^4 \frac{k_q}{k_{bT}} \qquad (71)$$

Similarly, by considering the membrane part of the load parameter in "(70)" to be zero, the small theorem load parameter is obtained as:

$$\frac{q}{E}\left(\frac{a}{t}\right)^4 = \frac{(A/t)}{12(1-\mu^2)}\frac{k_{bT}}{k_q} \qquad (70)$$

The parts of "(64)," to "(68)," that contains delta (Δ =A/t) are the membrane parts. Considering them zeros makes them (equation 64 to 68) to be those of small deflection theory.

3 Numerical Example

Analyze an ssss plate with large deflection that carries uniformly distributed load. The Poisson's ratio and Young's elastic modulus of the plate material are 0.316 and 200kN/mm² respectively. Span and thickness of the plate are a = 500mm and t = 5mm. The deflection of the plate is represented in trigonometric form as:

$$w = Ah = (Sin\,\pi R)\,(Sin\,\pi Q); \quad h^2 = (Sin^2\pi R)(Sin^2\pi Q)$$

Using this deflection function, the following stiffness coefficients are obtained:

$$k_{bRR} = \frac{\pi^4}{4};\ k_{bRQ} = \frac{\pi^4}{4};\ k_{bQQ} = \frac{\pi^4}{4};\ k_{mRR} = \frac{9\pi^4}{64}$$

$$k_{mRQ} = \frac{9\pi^4}{64};\ k_{mQQ} = \frac{9\pi^4}{64};\ k_{bT} = \frac{\pi^4}{4}\left(1 + \frac{2}{\beta^2} + \frac{1}{\beta^4}\right)$$

$$k_{mT} = \frac{9\pi^4}{64}\left(1 + \frac{2}{\beta^2} + \frac{1}{\beta^4}\right);\ k_q = \frac{4}{\pi^2};\ \frac{k_{bT}}{k_{mT}} = \frac{16}{9};\ \frac{k_q}{k_{mT}} = \frac{256}{9\pi^6\left(1 + \frac{2}{\beta^2} + \frac{1}{\beta^4}\right)};\ \theta = \frac{8}{3};\ c = -\frac{460.8737}{\pi^6\left(1 + \frac{2}{\beta^2} + \frac{1}{\beta^4}\right)} * \frac{q}{E}\left(\frac{a}{t}\right)^4;\ \frac{k_{bT}}{k_q} = \frac{\pi^6}{16}\left(1 + \frac{2}{\beta^2} + \frac{1}{\beta^4}\right);\ \frac{k_{mT}}{k_q} = \frac{9\pi^6}{256}\left(1 + \frac{2}{\beta^2} + \frac{1}{\beta^4}\right)$$

Values of shape functions and its derivatives at points whose coordinates are (0.5, 0.5) and (0.25, 0.25) are presented on Table 1.

Table 1: Values of shape functions and their derivatives

R	Q	dh/Dr	dh/dQ	d²h/dR²	d²h/dQ²	d²h/dRdQ	dh²/dR	dh²/dQ
0.5	0.5	0	0	9.8696	9.8696	0	0	0
0.25	0.25	1.5708	1.5708	4.9348	4.9348	4.9348	1.5708	1.5708
0	0	0	0	0	0	9.8696	0	0

4 Results and Discussions

Result from the present study was compared with that from Samuel Levy as presented on Table 2. The critical deflection below which the load from small deflection theory is approximately the same with the load from large deflection theory was determined using Table 3. The significance percentage difference between the load parameter from small and large deflection theories is 4.5%. The critical deflection whose load parameter is just less than 4.5% is w/t ≤ 1/3. This is obtained from Table 3. In-plane displacement along x-axis is presented on Table 4. This displacement was measured at the coordinate R = 0.25, Q = 0.25 and S = 0.5. Comparison of the displacement using large and small deflection theories was conducted on the same Table by simple percentage difference. From Table 4 it obvious that the small deflection theory overestimates the x-directed in-plane displacement. As the ratio of deflection to thickness increases, the percentage difference makes quadratic increase. On Table 5 is presented the normal stress along x direction, σx, which was measured at the two different coordinates: first coordinate is at R = 0.5, Q = 0.5 and S = 0.5 and second coordinate is at R = 0.25, Q = 0.25 and S = 0.5. For the first coordinate, it can be seen that the difference between the values from large and small deflection theories is zero for all the ratios of w/t. This implies that this difference is always zero at the center of ssss plate. However, the case is not the same at the second coordinate (0.25, 0.25, 0.5) where it is observed that the small deflection theory underestimates the normal stress along x-axis. Quadratic increase in the percentage difference between the large deflection and small deflection theories is observed as the ratio of w/t increases. It is pertinent to state here that for all w/t less than 0.35, the percentage difference is less than 6%. This implies that small deflection theory can be used at 95% level of accuracy to estimate the normal stress along x-axis for a rectangular plate so long as w/t is less than 0.35. From Table 6 it is seen that maximum x-y plane shear stress occurs at the coordinate: R = 0, Q = 0 and S = 0.5. That is at the corners of the

International Journal of Scientific & Engineering Research, Volume 11, Issue 3, March-2020
ISSN

plate. At this coordinate, the percentage difference between the shear stresses predicted by both the large and small deflection theories is zero for values of w/t ratio. It is also observed that

5 Conclusion

Based on the findings herein, it is obvious that for all values w/t less than 0.35, most parameters calculated using both large and small deflection theories differ in percentage by less than 7%. Hence, drawing a hypothesis that small deflection theory can be used to analyze thin rectangular ssss plate at high accuracy level for all cases of w/t less than 0.35. When the value of w/t is more than 0.35, analysis from small deflection theory shall not be reliable. Thus, large deflection theory is recommended for analysis of ssss plate when the value of w/t is up to or more than 0.35.

Table 2: Deflection for given values of load parameter

$qL^4/(Ebt^4)$	w/t	Center deflection, Present	Center deflection, Samuel	% Diff
0	0	0	0	0
12.1	0.497597	0.498	0.486	2.39
29.4	0.974384	0.974	0.962	1.29
56.9	1.439239	1.439	1.424	1.07
99.4	1.899036	1.899	1.87	1.55
161	2.352629	2.353	2.307	1.98
247	2.806981	2.807	2.742	2.37
358	3.247378	3.247	3.174	2.31
497	3.678012	3.678	3.6	2.17

Table 3: Load parameter for given values of Deflection for small and large deflection theorems

w/t	A/t	$qL^4/(Ebt^4)$ LDT	$qL^4/(Ebt^4)$ SDT	%diff
0	0	0	0.000	0.00
0.017	0.017	0.378	0.378	0.00
0.033	0.033	0.735	0.734	0.14
0.05	0.05	1.114	1.113	0.09
0.067	0.067	1.493	1.491	0.13
0.083	0.083	1.852	1.847	0.27
0.1	0.1	2.233	2.225	0.36
0.117	0.117	2.617	2.603	0.54
0.133	0.133	2.979	2.959	0.68
0.15	0.15	3.366	3.338	0.84
0.167	0.167	3.755	3.716	1.05
0.183	0.183	4.123	4.072	1.25
0.2	0.2	4.517	4.450	1.51
0.217	0.217	4.914	4.828	1.78
0.233	0.233	5.290	5.184	2.04
0.25	0.25	5.693	5.563	2.34
0.267	0.267	6.100	5.941	2.68
0.283	0.283	6.486	6.297	3.00
0.3	0.3	6.901	6.675	3.39
0.317	0.317	7.319	7.054	3.76
0.333	0.333	7.718	7.410	4.16
0.35	0.35	8.146	7.788	4.60
0.367	0.367	8.579	8.166	5.06
0.383	0.383	8.991	8.522	5.50
0.4	0.4	9.434	8.900	6.00
0.417	0.417	9.884	9.279	6.52
0.433	0.433	10.312	9.635	7.03
0.45	0.45	10.773	10.013	7.59
0.467	0.467	11.241	10.391	8.18
0.484	0.484	11.715	10.769	8.78

Legend: LDT means large deflection theorem; SDT means small deflection theorem

Table 4: X-axis displacement, u of the plate at coordinate (0.25, 0.25, 0.5)

	LDT	SDT	
w/t	u(0.25, 0.25, 0.5)	u(0.25, 0.25, 0.5)	%Diff
0	0	0	0
0.01	-0.00039	-0.00039	-0.33
0.02	-0.00078	-0.00079	-0.67
0.03	-0.00117	-0.00118	-1.01
0.04	-0.00155	-0.00157	-1.35
0.05	-0.00193	-0.00196	-1.69
0.1	-0.00380	-0.00393	-3.45
0.15	-0.00560	-0.00589	-5.26
0.2	-0.00733	-0.00785	-7.14
0.25	-0.00900	-0.00982	-9.09
0.3	-0.01060	-0.01178	-11.11
0.35	-0.01214	-0.01374	-13.21
0.4	-0.01361	-0.01571	-15.38
0.45	-0.01502	-0.01767	-17.65
0.5	-0.01636	-0.01963	-20.00

Table 5: X-axis normal stress, σ_x (N/mm²) of the plate at coordinates (0.5, 0.5, 0.5) and (0.25, 0.25, 0.5)

	LDT	SDT		LDT	SDT	
w/t	σ_x(0.5, 0.5, 0.5)	σ_x(0.5, 0.5, 0.5)	%Diff	σ_x(0.25, 0.25, 0.5)	σ_x(0.25, 0.25, 0.5)	%Diff
0	0	0	0	0	0	0
0.01	1.44	1.44	0	0.72	0.72	0.17
0.02	2.89	2.89	0	1.45	1.44	0.33
0.03	4.33	4.33	0	2.18	2.16	0.50
0.04	5.77	5.77	0	2.91	2.89	0.66
0.05	7.21	7.21	0	3.64	3.61	0.83
0.1	14.43	14.43	0	7.33	7.21	1.64
0.15	21.64	21.64	0	11.09	10.82	2.44
0.2	28.86	28.86	0	14.91	14.43	3.23
0.25	36.07	36.07	0	18.79	18.04	4.00
0.3	43.29	43.29	0	22.73	21.64	4.76
0.35	50.50	50.50	0	26.72	25.25	5.51
0.4	57.72	57.72	0	30.78	28.86	6.25
0.45	64.93	64.93	0	34.90	32.47	6.98
0.5	72.15	72.15	0	39.08	36.07	7.69

Table 6: X-Y plane shear stress, τ_{xy} (N/mm²) of the plate at coordinates (0.5, 0.5, 0.5) and (0.25, 0.25, 0.5)

	LDT	SDT		LDT	SDT	
w/t	τ_{xy}(0, 0, 0.5)	τ_{xy}(0, 0, 0.5)	%Diff	τ_{xy}(0.25, 0.25, 0.5)	τ_{xy}(0.25, 0.25, 0.5)	%Diff
0	0	0	0	0	0	0
0.01	-0.7500	-0.7500	0	-0.37436	-0.37498	-0.17
0.02	-1.4999	-1.4999	0	-0.74747	-0.74997	-0.33
0.03	-2.2499	-2.2499	0	-1.11933	-1.12495	-0.50
0.04	-2.9999	-2.9999	0	-1.48994	-1.49994	-0.67
0.05	-3.7498	-3.7498	0	-1.8593	-1.87492	-0.84
0.1	-7.4997	-7.4997	0	-3.68735	-3.74985	-1.69
0.15	-11.2495	-11.2495	0	-5.48416	-5.62477	-2.56
0.2	-14.9994	-14.9994	0	-7.24971	-7.4997	-3.45
0.25	-18.7492	-18.7492	0	-8.98401	-9.37462	-4.35

International Journal of Scientific & Engineering Research, Volume 11, Issue 3, March-2020
ISSN

6 REFERENCES

[1] A. G. Razdolsky, "Large Deflections of Elastic Rectangular Plates," International Journal for Computational Methods in Engineering Science and Mechanics, Volume 16, Issue 6, Pages 354-361. https://doi.org/10.1080/15502287.2015.1105882, 2015.

[2] E. Byklum and J. Amdahl,"A simplified method for elastic large deflection analysis of plates and stiffened panels due to local buckling," Thin-Walled Structures 40 (11), 925–953, 2002.

[3] F. Bloom and D. Coffin," Thin Plate Buckling and Postbuckling,". London: Chapman & Hall/CRC. ISBN 1-58488-222-0, 2001.

[4] H. Tanriöver and E.Senocak, "Large deflection analysis of unsymmetrically laminated composite plates: analytical–numerical type approach", International Journal of Non-linear Mechanics. Vol. 39, pp. 1385–1392, 2004.

[5] I. Shufrin, O. Rabinovitch, and M. Eisenberger, "A semi-analytical approach for the nonlinear large deflection analysis of laminated rectangular plates under general out-of-plane loading,"International Journal of Non-Linear Mechanics 43, 328–340, 2008.

[6] J. I. ENEM, ''Geometrically Nonlinear Analysis of Isotropic Rectangular Thin Plates Using Ritz Method''. PhD Thesis: Department of Civil Engineering Federal University of Technology, Owerri Nigeria, 2018.

[7] Lei Zhang • Jizeng Wang • You-He Zhou," Wavelet solution for large deflection bending problems of thin rectangular plates," Arch Appl Mech, 85: pp.355–365, DOI 10.1007/s00419-014-0960-9, 2014.

[8] Manuel Stein, ''Analytical results for postbuckling behavior ofplates in compression and in shear," National Aeronautics and Space Administration (NASA). NASA Technical Memorandum 85766, 1984.

[9] M. R. Elsami, ''Buckling and Postbuckling of Beams, Plates, and Shells," Springer International Publishing AG. ISBN 978-3-319-62368-9, 2018.

[10] Mohammad Rezaiee-Pajand, Hossein Estiri,"A Comparison of Large Deflection Analysis of Bending Plates by Dynamic Relaxation," Periodica Polytechnica Civil Engineering, 60(4), pp. 619–645, DOI: 10.3311/PPci.8728, 2016.

[11] O. A. Oguaghamba, ''Analysis of buckling and postbuckling loads of Isotropic thin rectangular plates''. PhD Thesis: Department of Civil Engineering Federal University of Technology, Owerri Nigeria, 2015.

[12] O. M. Ibearugbulem, Lecture Note on "Large deflection analysis of continuums," Department of Civil Engineering. Federal University of Technology, Owerri, Nigeria. Unpublished, 2017.

[13] P. D. Onodagu, ''Nonlinear Dynamic Analysis of Thin Rectangular Plates using Ritz Method''. PhD Thesis: Department of Civil Engineering Federal University of Technology, Owerri Nigeria, 2018.

[14] S.Levy,"Bending of rectangular plates with large deflections,"Technical notes: National Advisory Committee for Aeronautics (NACA), N0. 846, 1942.

[15] S. Levy, "Square plate with clamped edges under normal pressure producing large deflections," Technical notes: National Advisory Committee for Aeronautics (NACA), N0. 740, 1942.

[16] S. A. M.GhannadPour and M. M.Alinia, ''Large deflection behavior of functionally graded plates under pressure loads, Composite Structures," Vol. 75, pp.67–71, 2006.